

1. Each of three cards has an integer written on it. The three integers p, q, r satisfy the condition $0 < p < q < r$. Three players A, B, C mix the cards and pick one each. They record the number on their card, mix the cards and pick one each again. The number on the card they select is added to their previous number. This process is repeated at most ten times, after which A has 20 points, B has 10 points, and C has 9 points. If B got the r card in the last round, determine, with proof,

Solutions

1. Because a constant number of points, $p + q + r$, is awarded in each round, $p + q + r$ must divide the total number of points, 39.

(b) Since $\triangle AOD \sim \triangle BCO$, $\frac{AD}{OB} = \frac{AO}{BC}$, or $(AD)(BC) = (AO)(OB)$. Since $AO = OB = \frac{1}{2}(AB)$, we get $(AB)^2 = 4(AD)(BC)$.

4. Suppose we could write a number both as $n(n+1)$ and as $m(m+1)(m+2)(m+3)$, where n and m are positive integers. Multiplying the outer and inner factors of the second expression gives $(n^2+3)(n^2+3+2)$. Letting $k = n^2+3$, we have $(k+1) = (k+2)$. Adding 1 to each side, $k^2+k+1 = k^2+2k+1 = (k+1)^2$. So k^2+k+1 is a perfect square. But $k^2 < k^2+k+1 < (k+1)^2 = k^2+2k+1$. Thus, k^2+k+1 is a perfect square that lies between 2 consecutive perfect squares, which is impossible. Therefore, there are no integers that can be written both as the product of two and also four consecutive positive integers.

5. Method 1

In $\triangle ABC$, since CD bisects $\angle ACB$, $\frac{AD}{DB} = \frac{AC}{BC} = \frac{10}{6}$,
from which $AD = 4$ and $DB = 2$.

Using the Law of Cosines on $\triangle ABC$,

$$5^2 = 10^2 + 6^2 - 2(10)(6) \cos A \quad \cos A = \frac{37}{40}.$$

Using the Law of Cosines on $\triangle ADC$,

$$CD^2 = 10^2 + 4^2 - 2(10)(4) \frac{37}{40} \quad CD = \sqrt{42}.$$

Using the Law of Cosines on $\triangle ADC$ again,

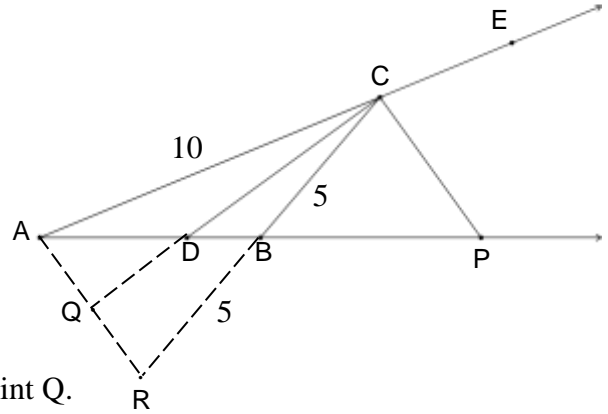
$$10^2 = 4^2 + (\sqrt{42})^2 - 2(4)(\sqrt{42}) \cos \angle ADC \quad \cos \angle ADC = \frac{-\sqrt{42}}{8}.$$

Since $\angle CDP$ is supplementary to $\angle ADC$,

Method 2

From Method 1, $AD = 4$ and $DB = 2$, and $\angle DCP$ is a right angle.

Extend CB its own length through B to a point R , and construct AR . Extend CD through D to meet AR at point Q .



Since $CR = 10$, $\triangle ACR$ is isosceles, and since CQ bisects vertex angle ACR , CQ is also an altitude in $\triangle ACR$. Then AR is parallel to CP since CQ is perpendicular to both. Therefore, $\angle BCP$ and $\angle BRA$ are congruent alternate interior angles.

Then, $\triangle ABR \cong \triangle PBC$, so that $BP = AB = 6$, and $DP = DB + BP = 8$.

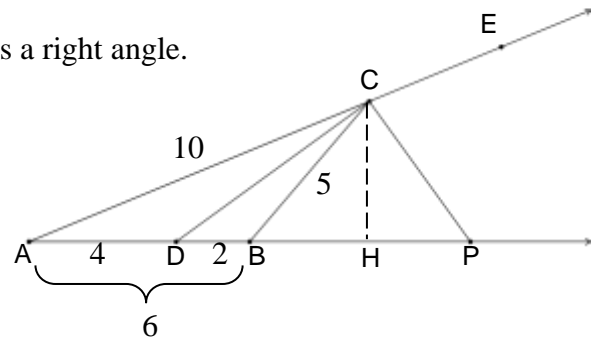
Method 3

From Method 1, $AD = 4$ and $DB = 2$, and $\angle DCP$ is a right angle.

Construct the altitude of $\triangle ABC$ from C , meeting Line AB at point H . Let $BH = x$ and $CH = h$.

Using the Pythagorean Theorem on $\triangle ACH$,
 $(x + 6)^2 + h^2 = 10^2$ $h^2 = 64 - 12x + x^2$

Using the Pythagorean Theorem on $\triangle DCH$,
 $(x + 2)^2 + h^2 = 5^2$ $h^2 = 21 - 4x + x^2$.



Setting the two equations equal and solving for x , we get $x = \frac{13}{4}$, from which $h = \frac{\sqrt{231}}{4}$.

Using the Pythagorean Theorem on $\triangle CDH$, $CD = \sqrt{42}$.

Because right $\triangle DCP$ and right $\triangle DHC$ share $\angle D$, they are similar. Then

$$\frac{DH}{DC} = \frac{DC}{DP} \quad DP = \frac{(DC)^2}{DH} = \frac{42}{\frac{21}{4}} = 8.$$