



1. Let us say that a person is “acquainted” with another person if each of them knows the other. Suppose there is a gathering of 100 people and that among every four people at the gathering there is at least one who is acquainted with the other three. Prove that in this gathering, there is at least one person who is acquainted with all the remaining 99 people.

2. Suppose that a and b are integers such that $a^2 + 2$ and $b^2 + 2$ are squares. Prove that a and b are each multiples of 3.

3. Let $P(x)$ be a polynomial with integer coefficients.
 - (i) Prove that it is not possible for $P(7) = 11$ and $P(11) = 13$.
 - (ii) Suppose that $P(0) = P(1) = 1$. Prove that the equation $P(x) = 0$ has no integer solutions.

- 4.

1. Pick any person, say B. If B is acquainted with all the others, we are done.

Otherwise, there is a person C such that B is not acquainted with C. Then C is not acquainted with

4. Method 1 (elegant and simple!!)

We must show that quadrilateral ABCD is cyclic.

The lines intersect on the x and y-axes as shown, and the axes bisect the angles formed. Represent the origin as point T, and the intersections of

and with the and -axes as S and R, respectively.

Let $\angle APR = \angle RPB$ and $\angle BQT = \angle TQC$.

Since $\angle DAR$ is an exterior angle of $\triangle APR$, we have $m\angle DAR = \angle APR + m\angle ARP$.

Also, $m\angle ARP = m\angle TRQ$. Since $\triangle TRQ$ is a right triangle,

$m\angle TRQ = 90 - m\angle RQT$ & $m\angle ARP = 90 - \angle$. Therefore, $m\angle DAR = \angle APR + 90 - \angle$.

The common solution to these two equations (the hypothetical center of the circle) is

$$P \left(\frac{8}{63}, \frac{40}{63} \right)$$

Now, rewriting the coordinates of points A, B, C, and D with a common denominator 63,

$$A \left(\frac{56}{63}, \frac{140}{63} \right), B \left(\frac{72}{63}, \frac{108}{63} \right), C \left(\frac{168}{63}, \frac{84}{63} \right), \text{ and } D \left(\frac{216}{63}, \frac{180}{63} \right),$$

and using the distance formula to find the lengths \overline{AP} , \overline{BP} , \overline{CP} , and \overline{DP} , we obtain:

$$AP = \sqrt{\frac{16^2}{63^2} + \frac{(1208)^2}{63^2}}, \quad BP = \sqrt{\frac{112^2}{63^2} + \frac{176^2}{63^2}}, \quad CP = \sqrt{\frac{208^2}{63^2} + \frac{(116)^2}{63^2}}, \quad DP = \sqrt{\frac{176^2}{63^2} + \frac{112^2}{63^2}}.$$

Since $16^2 + 208^2 = 43520 = 112^2 + 176^2$, A, B, C, and D are all the same distance from P.

A 24 720 25, Let B, C.