THE 2013-2014 KENNESAW STATE UNIVERSITY HIGH SCHOOL MATHEMATICS COMPETITION PART II

In addition to scoring student responses based on whether a solution is correct and complete, consideration will be given to elegance, simplicity, originality, and clarity of presentation.

Calculators are <u>NOT</u> permitted.

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1. A and B both represent nonzero digits (not necessarily distinct). If the base ten numeral $\underline{A}\,\underline{B}$ divides, without remainder, the base ten numeral $\underline{A}\,\underline{0}\,\underline{B}$ (whose middle digit is zero), find, with proof, all possible values of $\underline{A}\,\underline{B}$.

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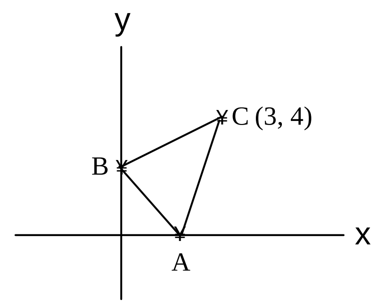
2. A and B are points on the positive x and positive y axes respectively and C is the point with coordinates (3, 4). Prove that the perimeter of triangle ABC is greater than 10.



3. One solution for the equation $a^2 + b^2 + c^2 + 2 = abc$ is $a = 3$, $b = 3$ and $c = 4$.

   a. Find a solution (a, b, c) where a, b, and c are integers all larger than 10.

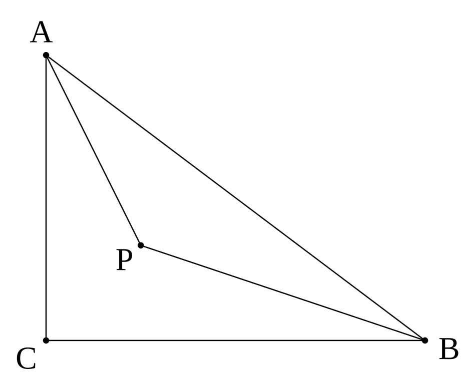
   b. Prove that there are infinitely many solutions (a, b, c) where a, b, and c are positive integers.

4. Consider the equation $\sqrt{x} = \sqrt{a} + \sqrt{b}$, where x is a positive integer.

   a. Prove that the equation has a solution (a, b) where a and b are both positive integers, if and only if x has a factor which is a perfect square greater than 1.

   b. If x ! 1,000, compute, with proof, the number of values of x for which the equation has at least one solution (a, b) where a and b are both positive integers.

5. In right triangle ABC, AC = 6, BC = 8 and AB = 10. PA and PB bisect angles A and B respectively. Compute, with proof, the ratio $\frac{PA}{PB}$.



1. Of course, this problem can be done by trial and error (there are only 81 possibilities), but we present a more elegant solution.

Suppose $\overline{AB}$

3. Suppose we begin with two positive integers a and b, and we try to find a third integer x such that $a^2 + b^2 + x^2 + 2 = abx$. Then the problem can be thought of as finding an integer solution (if one exists) for the quadratic equation $x^2 - (ab)x + (a^2 + b^2 + 2) = 0$.

If there is some integer solution x = c, then there must exist a real number d such that

$$x^2 - (ab)x + (a^2 + b^2 + 2) = (x - c)(x - d) = x^2 - (c + d)x + cd$$

Comparing the coefficients on the left and right sides of this last equation, we know that ab = c + d, so that d = ab – c is also an integer. Therefore, given any three integers a, b, and c such that $a^2 + b^2 + c^2 + 2 = abc$, we can replace c with ab – c to obtain another solution.

We know that (4, 3, 3) is a solution. So we can replace one of the 3's with 4 · 3 – 3 = 9 to get the solution (4, 3, 9). Since a, b, and c are interchangeable, We can obtain other solutions by repeatedly replacing the smallest number (which we will call c) by ab – c. Hence, listing the numbers in decreasing order at each step, we obtain the following solutions:

(4, 3, 3) → (9, 4, 3) → (33, 9, 4) → (293, 33, 9) → (9660, 293, 33).

Since this process can be repeated indefinitely, there are infinitely many positive integer solutions (a, b, c) to the given equation.

4. (i) Given $\sqrt{x} = \sqrt{a} + \sqrt{b}$.

Suppose $x = k^2 y$, with k and y positive integers, and k > 1. We must prove that there exists at least one pair of positive integers (a, b) that satisfies the equation.

We have $\sqrt{x} = \sqrt{k^2 y} = k\sqrt{y}$. Since k > 1, then $k - 1 > 0$. Therefore,

$$\sqrt{x} = k\sqrt{y} = (k-1)\sqrt{y} + \sqrt{y} = \sqrt{(k-1)^2 y} + \sqrt{y}.$$

Since both $(k-1)^2 y$ and y are both positive integers, setting $a = (k-1)^2 y$ and b = y gives the desired result.

5. <u>Method 1</u>:

We will refer to ∠CAB as ∠A and ∠CBA as ∠B.
So that m∠A + m∠B = 90°.

Then m∠P = 180 − ½ (m∠A + m∠B) = 135°.
So that, m∠PAB + m∠PBA = 45. Represent the measures of these two angles with θ and 45 − θ.

Using the Law of Sines on △APB

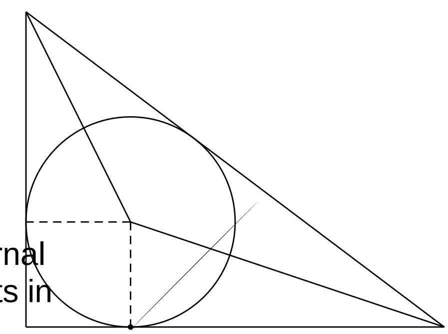
$$\frac{PA}{PB} = \frac{\sin(45 - \theta)}{\sin\theta} = \frac{\sin 45 \cos\theta - \cos 45 \sin\theta}{\sin\theta} = \sin 45 \cot\theta - \cos 45.$$

Now $\cot\theta = \cot(\frac{1}{2}A) = \frac{1+\cos A}{\sin A}$ (using the appropriate half-angle formula)

But in △ABC, $\cos A = \frac{6}{10}$ and $\sin A = \frac{8}{10}$, making $\cot\theta = \frac{1+\frac{6}{10}}{\frac{8}{10}} = 2$.

Finally, $\frac{PA}{PB} = (\sin 45)(2) - \cos 45 = \frac{\sqrt{2}}{2}(2) - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$.

<u>Method 2</u>:



Note that since point P is the intersection of the angle bisectors of △ABC, P is the incenter (the center of the inscribed circle).

Noting that the tangent segments to a circle from an external point are congruent, represent the lengths of the segments in the diagram as shown.

Then 6 − x + 8 − x = 10 and x = 2.
Therefore, right △ARP has side lengths 2, 4, and $2\sqrt{5}$, and right △BMP has side lengths 2, 6, and $2\sqrt{10}$.

Therefore, $\frac{PA}{PB} = \frac{2\sqrt{5}}{2\sqrt{10}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.