



February 7, 2007
Exam Time ± 2 Hours

NO CALCULATORS

1. Prove that $x^2 + y^2 = A^3$ always has integer solutions (x, y) whenever A is a

1. Since $x^2 - y^2 = (x - y)(x + y) = A^3$ and $x - y = A^2$ and $x + y = A$
 Adding both of these equations, we obtain $2x = A^2 + A$ and $x = \frac{A(A + 1)}{2}$.
 Subtracting the two equations, we obtain $2y = A^2 - A$ and $y = \frac{A(A - 1)}{2}$.
 For any positive integer A , both $A(A-1)$ and $A(A+1)$ are products of 2 consecutive integers and are, therefore, both even. Hence, $x = \frac{A(A + 1)}{2}$ and $y = \frac{A(A - 1)}{2}$ are a pair of integer solutions to the equation.

2. The circumcenter of a triangle is the intersection of the perpendicular bisectors of its sides, namely, points E, F, G, and H. Any point on the perpendicular bisector of a segment is equidistant from the segment's endpoints. Since G is on the perpendicular bisector of \overline{AD} and \overline{DC} , $\overline{AG} = \overline{DG} = \overline{CG}$. Since E is on the perpendicular bisector of \overline{AB} and \overline{BC} , $\overline{AE} = \overline{EB} = \overline{EC}$. Since $\overline{AG} = \overline{CG}$ and $\overline{AE} = \overline{EC}$ points E and G are both equidistant from points A and C. Therefore, \overline{EG} lies along the perpendicular bisector of \overline{AC} . Since point P lies on \overline{EG} , it is also on perpendicular bisector of \overline{AC} , making P equidistant from A and C. Thus $\overline{AP} = \overline{CP}$. In a similar manner, we can prove $\overline{BP} = \overline{DP}$.

3. Since $P(a) = P(b) = P(c) = P(d) = 4$, the polynomial $f(x) = P(x) - 4$ has $a, b, c,$ and d as zeros. Therefore, $f(x) = P(x) - 4 = (x - a)(x - b)(x - c)(x - d)g(x)$, where $g(x)$ is a polynomial with integer coefficients.

Suppose there exists an integer m such that $P(m) = 7$. Then

$$f(m) = P(m) - 4 = 7 - 4 = 3 = (m - a)(m - b)(m - c)(m - d)g(m)$$

5. Let such a sequence be $(m + 1), (m + 2), \dots, (m + k)$. The sum of the terms can be written as

$$km + \frac{k(k-1)}{2} = \frac{k(2m+k+1)}{2}.$$

Thus $\frac{k(2m+k+1)}{2} = 2007 = 3^2 \cdot 223$ and $k(2m+k+1) = 18 \cdot 223$.

Since 223 is prime and is a factor of the product $k(2m+k+1)$, then 223 divides either k or $2m+k+1$. Since $1+2+3+\dots+63 = 2016 > 2007$, then $k < 63$. Therefore, 223 divides $2m+k+1$ and k must be a factor of 18. Thus, $k = 18, 9, 6, 3, 2$, or 1.

If $k = 18$, then $2m+k+1 = 223$ implies $m = 102$. This gives the sequence
103, 104, 105, . . . , 120.

If $k = 9$, then $2m+k+1 = 2 \cdot 223 = 446$, and $m = 218$. This gives the sequence
219, 220, 221, . . . , 227.

If $k = 6$, then $2m+k+1 = 3 \cdot 223$ implies $m = 331$. This gives the sequence
332, 333, 334, . . . , 337.

If $k = 3$, then $2m+k+1 = 6 \cdot 223 = 1338$, and $m = 667$. This gives the sequence
668, 669, 670.

If $k = 2$, then $2m+k+1 = 9 \cdot 223$ implies $m = 1002$. This gives the sequence
1003, 1004.

If $k = 1$, then $2m+k+1 = 18 \cdot 223 = 4014$ implies $m = 2006$. This gives the sequence with one number, 2007, which we are asked not to include.

Thus, there are five sequences of consecutive positive integers whose terms sum to 2007:

103, 104, 105, . . . , 120
219, 220, 221, . . . , 227
332, 333, 334, . . . , 337
668, 669, 670
1003, 1004